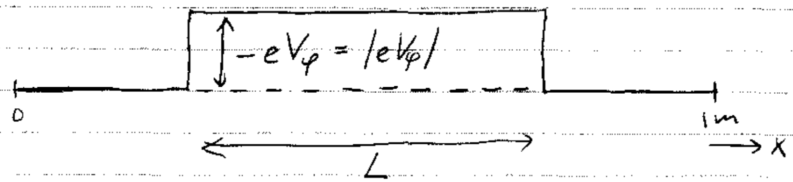
# Uitwerking Tentamen Kwantum Fysica I 24 aug 2005

## Problem 1

a) electron charge is $-e = -|e|$
Potential $-eV_\varphi > 0$ for $V_\varphi < 0V$.




b) $\lambda = \frac{h}{p}$ , $p = \sqrt{2mE_{kin}} = \sqrt{2mE_{k0}}$

$h = 6.626 \cdot 10^{-34}$ Js $m = 9.1 \cdot 10^{-31}$ $E_{kin} = 5000\,eV$

$\Rightarrow \lambda = 1.7 \cdot 10^{-11}\,m = \frac{h}{\sqrt{2mE_{k0}}}$

c) Inside the phase controller $E_{kin} = E_{k0} - U$

with $U = -eV_\varphi \Rightarrow$

$E_{kin} = E_{k0} + eV_\varphi \Rightarrow p = \sqrt{2m(E_{k0} + eV_\varphi)}$

$\Rightarrow \lambda = \frac{h}{\sqrt{2m(E_{k0} + eV_\varphi)}}$



d) The phase evolution of an electron while traversing $L$ is:

$\Delta\varphi_L = \frac{2\pi L}{\lambda}$

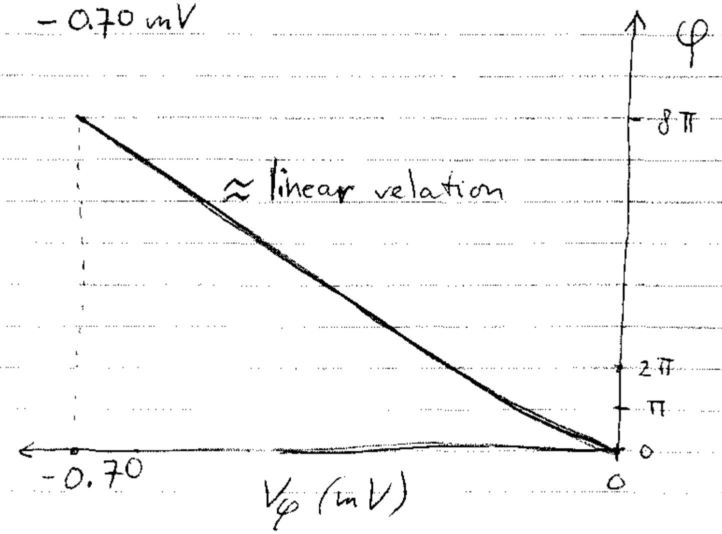
In the phase controller $\lambda$ is a function of $V_\varphi$ (see c)). The resulting control phase $\varphi$ (the phase difference with the other beam) is

$$\varphi = \Delta\varphi_L(V_\varphi = 0) - \Delta\varphi_L(V_\varphi < 0)$$

$$= \frac{2\pi L}{\lambda(V_\varphi=0)} - \frac{2\pi L}{\lambda(V_\varphi<0)} = \frac{2\pi}{h}\left(L\sqrt{2mE_{k0}} - L\sqrt{2m(E_{k0}+eV_\varphi)}\right)$$

Solving this for $\varphi = 0, \pi, 2\pi, 8\pi$ gives

| $\varphi$ | $V_\varphi$ |
|---|---|
| 0 | 0 mV |
| $\pi$ | -0.09 mV |
| $2\pi$ | -0.18 mV |
| $8\pi$ | -0.70 mV |



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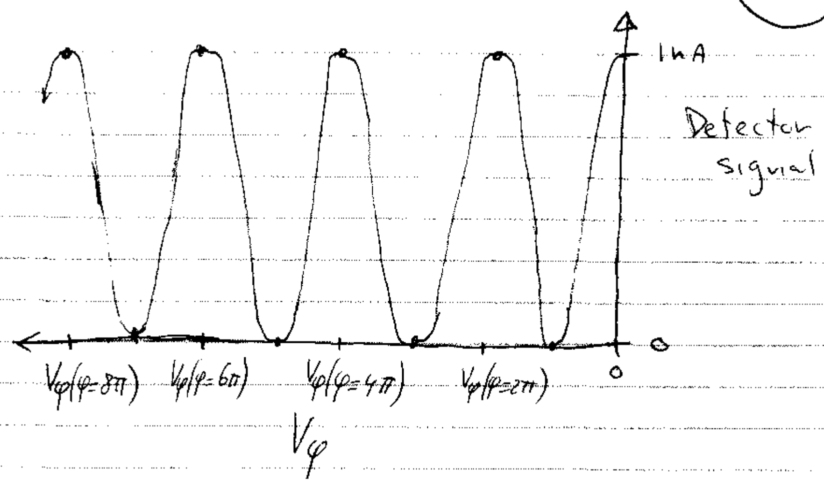
e)




f) The transmission coefficient $T$ represent the probability that an electron is transmitted through the phase controller. So, if this probability is $P_T = T$, then the wavefunction amplitude is reduced by a factor $\sqrt{P_T} = \sqrt{T}$. Since interference at the detector is the result of interfering amplitudes of the wavefunction, we should analyze the effect of a reduced $T$ based on $\sqrt{T}$.

| $\varphi$ | $T$ | $\sqrt{T}$ |
|---|---|---|
| 0 | 1 | 1 |
| $2\pi$ | $3/4$ | $\sqrt{3/4}$ |
| $4\pi$ | $1/2$ | $\sqrt{1/2}$ |
| $6\pi$ | $1/4$ | $\frac{1}{2}$ |
| $8\pi$ | 0 | 0 |

estimated from the figure

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For the $\varphi$-values $0, 2\pi, 4\pi, 6\pi$ and $8\pi$ the effective phase difference is **zero**.
(constructive interference, maxima in the detector signal).
Immediately after coming out of the slits, the amplitudes of the two beams are equal, say $\Psi_0$:

For beam without phase controller $\Psi_1 = \Psi_0$
For beam with phase controller $\Psi_2 = \Psi_0$

Just before the detector this is then

For beam without phase detector $\Psi_1 \propto \Psi_0$
For beam with phase detector $\Psi_2 \propto \sqrt{T}\Psi_0$

The detector signal $S$ is proportional to

$$S \propto |\Psi_1 + \Psi_2| \propto |\Psi_0|^2 \, |1 + \sqrt{T}|^2$$

When normalized to the signal $S_0$ for $T=1$ and $\varphi = 0$, $\frac{S}{S_0}$ depends on $T$ as $\frac{1}{4}(1+\sqrt{T})^2$ for the maxima in the interference pattern

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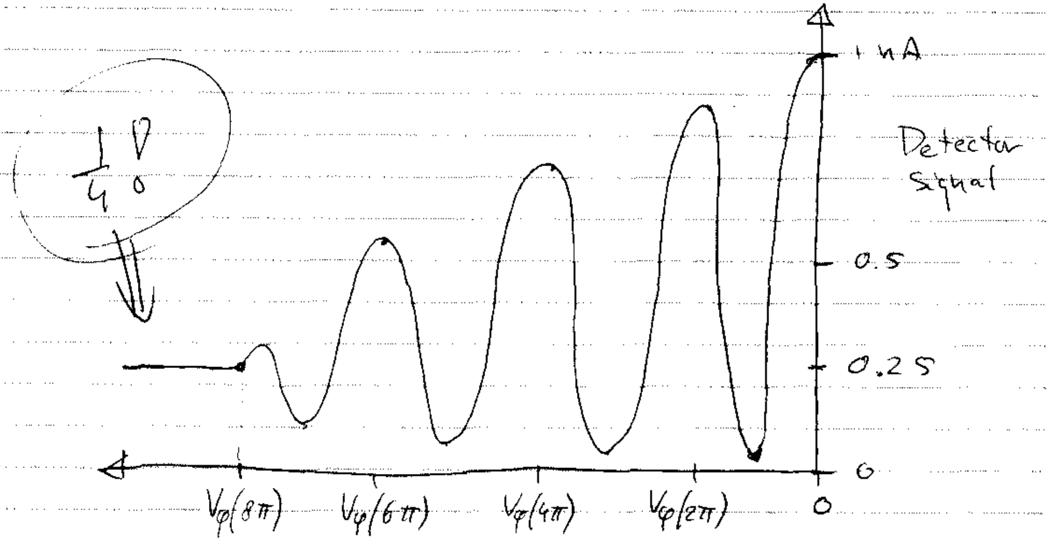
| $\varphi$ | $\frac{S}{S_0} = \frac{(1+\sqrt{T})^2}{4}$ | $S$ (nA) |
|---|---|---|
| 0 | 1 | 1 |
| $2\pi$ | 0.87 | 0.87 |
| $4\pi$ | 0.72 | 0.72 |
| $6\pi$ | 0.56 | 0.56 |
| $8\pi$ | 1/4 | 0.25 |

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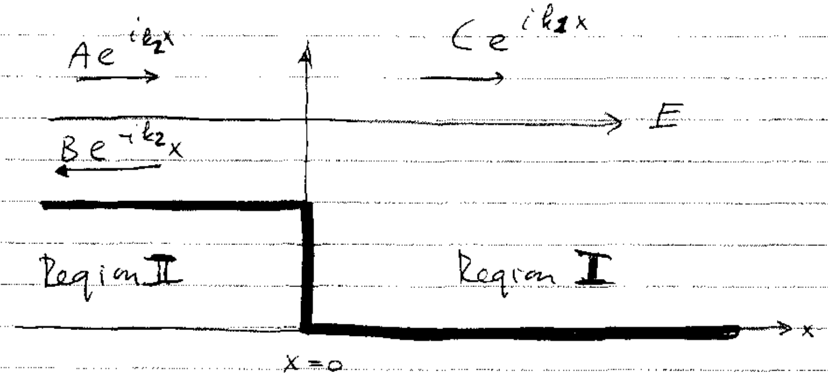
## Problem 2 (see book p. 222)

a)




$$\begin{cases} \text{Incoming wave} & Ae^{ik_2x} \\ \text{Reflected wave} & Be^{-ik_2x} \\ \text{Transmitted wave} & Ce^{ik_1x} \end{cases}$$

For notation, we made the (arbitrary) choice to call $x<0$ region II, and $x>0$ region I.

$\Rightarrow$ Schrödinger equation:

$$\begin{cases} \text{In region II} & -\frac{\hbar^2}{2m}\varphi_{xx} = (E-V)\varphi \\ \text{In region I} & -\frac{\hbar^2}{2m}\varphi_{xx} = E\varphi \end{cases} \Rightarrow$$

$$\begin{cases} \text{Define } \frac{\hbar^2 k_2^2}{2m} = E - V \ , \ \frac{\hbar^2 k_1^2}{2m} = E \Rightarrow \end{cases}$$

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$$\begin{cases} \varphi_{xx} = -k_2^2 \varphi & , \quad x<0 \\ \varphi_{xx} = -k_1^2 \varphi & , \quad x>0 \end{cases}$$

$$\begin{cases} \varphi_{II} = A e^{ik_2 x} + B e^{-ik_2 x} & , \quad x<0 \\ \varphi_{I} = C e^{ik_1 x} & , \quad x>0 \end{cases}$$

at $x=0$, $\begin{cases} \varphi(x) \text{ is continuous} \\ \varphi_x = \frac{d\varphi(x)}{dx} \text{ is continuous} \end{cases} \Rightarrow$

$A+B=C$ (from $\varphi(x)$ continuous)

$ik_2 A e^{ik_2 0} - ik_2 B e^{-ik_2 0} = ik_1 C e^{ik_1 0} \Rightarrow$

$A - B = \frac{k_1}{k_2} C$ (from $\varphi_x(x)$ continuous)

Now solve $\frac{C}{A}$ and $\frac{B}{A}$ (for example, normalize to incoming beam)

$$\Rightarrow \begin{cases} \dfrac{C}{A} = \dfrac{2}{1 + k_1/k_2} \\[2ex] \dfrac{B}{A} = -\dfrac{-1 + k_1/k_2}{1 + k_1/k_2} \end{cases}$$

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Transmission coefficient

$$T = \left|\frac{C}{A}\right|^2 = \frac{4}{\left(1 + k_1/k_2\right)^2}$$

$$R = \left|\frac{B}{A}\right|^2 = \left|\frac{-1 + k_1/k_2}{1 + k_1/k_2}\right|^2 \Rightarrow$$

$$R = \left|\frac{1 - \frac{k_2}{k_1}}{1 + \frac{k_2}{k_1}}\right|^2$$

b) R is the same for a step-up and step-down potential (see also fig 7.21 in book). This must be the case, since the Schrödinger equation describes time-evolution of a system in a deterministic way without losses $\Rightarrow$ Reversing the direction of time $(t \to -t)$ for a transmission event should not change the probabilities before and after the event.

## Problem 3

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a)

$$\hat{H}|\Psi_i\rangle = E_i|\Psi_i\rangle$$

b) During a transition between level $E_2$ and $E_1$, the system is in a superposition state $|\Psi\rangle = c_1|\Psi_1\rangle + c_2|\Psi_2\rangle$ with $|\Psi_1\rangle, |\Psi_2\rangle$ the energy eigenstates associated with $E_1$ and $E_2$.

Then, the dipole moment oscillates as

$$\langle\hat{D}\rangle(t) = \langle\Psi(t)|\hat{D}|\Psi(t)\rangle$$

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\Psi(t_0)\rangle, \text{ where}$$

$t_0$ some initial moment in time.

We can arbitrarily define $t_0 = 0$ and $|\Psi(t=t_0)\rangle$ as $|\Psi_0\rangle$, and $|\Psi_0\rangle = c_1|\Psi_1\rangle + c_2|\Psi_2\rangle$

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Then, $\langle\Psi(t)|\hat{D}|\Psi(t)\rangle =$

$$\left(\langle\Psi_0|e^{+\frac{i}{\hbar}\hat{H}t}\right)\hat{D}\left(e^{-\frac{i}{\hbar}\hat{H}t}\right)|\Psi_0\rangle =$$

$$\left(c_1^* e^{\frac{i}{\hbar}E_1 t}\langle\Psi_1| + c_2^* e^{\frac{i}{\hbar}E_2 t}\langle\Psi_2|\right)\hat{D}\left(c_1 e^{-\frac{i}{\hbar}E_1 t}|\Psi_1\rangle + c_2 e^{-\frac{i}{\hbar}E_2 t}|\Psi_2\rangle\right) =$$

$c_1^* c_1 \langle\Psi_1|\hat{D}|\Psi_1\rangle$ + (stationary)

$c_2^* c_2 \langle\Psi_2|\hat{D}|\Psi_2\rangle$ + (stationary)

$2\,\mathrm{Re}\left\{c_1^* c_2\, e^{-\frac{i(E_2-E_1)}{\hbar}t}\langle\Psi_1|\hat{D}|\Psi_2\rangle\right\}$

The last term is not zero since $[\hat{H},\hat{D}] \neq 0$, and oscillates in time at frequency

$$f = \frac{(E_2 - E_1)}{\hbar} \Rightarrow$$

$$hf = E_2 - E_1$$

It can only oscillate at this frequency, since the most general state of this system is in the form $|\Psi\rangle = c_1|\Psi_1\rangle + c_2|\Psi_2\rangle$ (only 2 levels)